25 years of SAT
Maximum Satisfiability for Real-World Optimization

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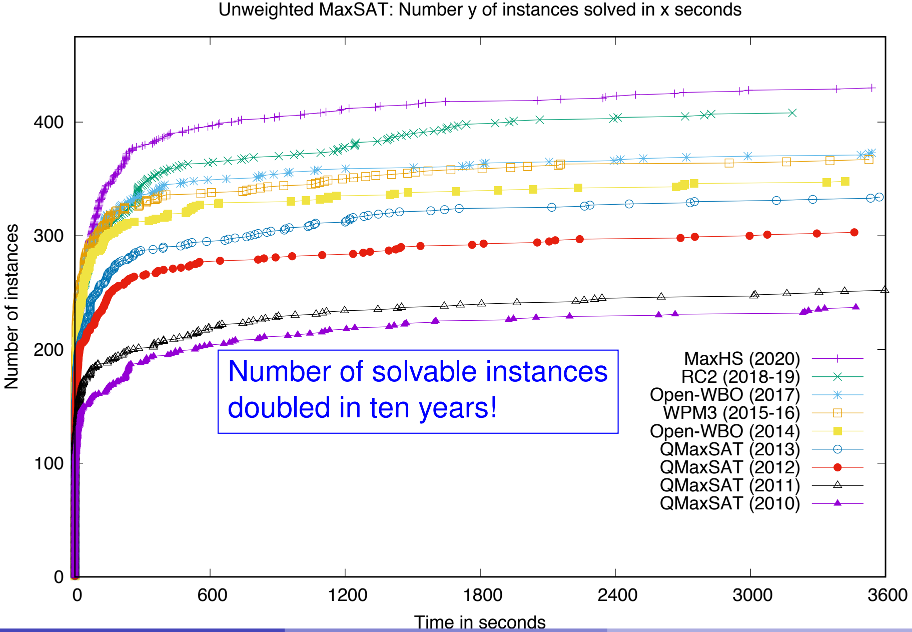
Haifa

Motivation: MaxSAT

- Competitive optimization paradigm.
  - planning, scheduling, data analysis, machine learning, knowledge representation and reasoning, verification, ...
- State-of-the-art complete solvers (for industrial instances) build on the success of CDCL SAT solvers.
  - Optimization task reduced into a sequence of satisfiability queries.
  - Extensive use of incremental SAT.
- New application domains and solver improvements annually.

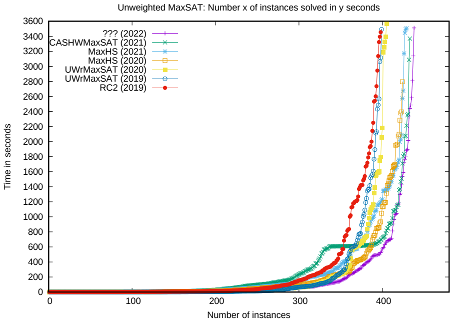
  (Recent survey in [Bacchus, Järvisalo, and Martins, 2021])

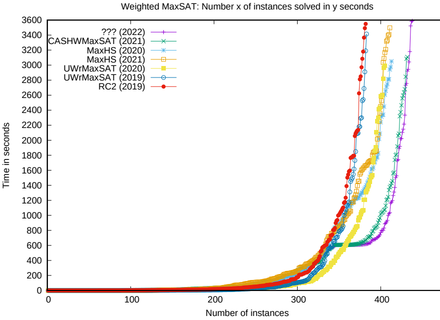
Solver Progression



Number of solvable instances doubled in ten years!

Solver Progression





Goals of the Talk

- Overview of existing *complete* SAT-based MaxSAT algorithms.
  - Solution Improving
  - Core Guided
  - Implicit Hitting Set
- Overview of the additional techniques employed by solvers.
- (If time) Other interesting MaxSAT-related research directions.

Thanks to Ruben Martins and Matti Järvisalo for contributions to the slides

Notation: Maximum Satisfiability

- Optimisation extension of Boolean Satisfiability (SAT)
- An instance:
  - a set of hard clauses,
  - a linear objective function *cost*
- Find τ that:
  - satisfies all hard clauses and
  - minimizes cost
- **Note:** All SAT-based solvers support weights.

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  - a linear objective function *cost*
  - equivalent to a set of (unit) soft clauses.
- Find τ that:
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- **Note:** All SAT-based solvers support weights.

$$\mathcal{F}_H = \{(b_1 \lor x), (\neg x \lor b_2), (b_2 \lor y), (\neg y, b_3)\}$$

$$cost \equiv b_1 + b_2 + b_3$$

$$\mathcal{F}_S = \{(\neg b_1), (\neg b_2), (\neg b_3)\}$$

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$$\boxed{\begin{array}{l}\tau(y) = \tau(b_1) = \tau(b_3) = 0 \\ \tau(x) = \tau(b_2) = 1\end{array}}$$

$$\mathcal{F}_H = \{(b_1 \vee x), (\neg x \vee b_2), (b_2 \vee y), (\neg y, b_3)\}$$

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$$cost(\tau) = 1$$

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$$\mathcal{F}_H = \{(b_1 \lor x), (\neg x \lor b_2), (b_2 \lor y), (\neg y, b_3)\}$$

$$cost \equiv w_1 b_1 + w_2 b_2 + w_3 b_3$$

Example Problem

Shortest Path

- find shortest path from S to G.
- one variable for each square.
  - true if path goes through square.
- (hard) clauses:
  - enforce correspondence between solutions and paths.
- cost function:
  - measures length of path.

n	o		p	q
h	i	j	k	**G**
c	d	e	l	r
a		f		s
S	b	g	m	t

Note: Best solved with other methods, used here to illustrate different MaxSAT algorithms.

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$$\text{VAR}(\mathcal{F}) = \{a, b, c, \ldots, r, s, t\}$$

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$$\textsc{var}(\mathcal{F}) = \{a, b, c, \ldots, r, s, t\}$$

$$\mathcal{F}_H = \textsc{IsPath}(S, G)$$

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$$\textsc{Var}(\mathcal{F}) = \{a, b, c, \ldots, r, s, t\}$$

$$\mathcal{F}_H = \textsc{IsPath}(S, G)$$

$$cost = a + b + c + \ldots + r + s + t$$
$$cost = \{(\neg a), (\neg b), \ldots, (\neg s), (\neg t)\}$$

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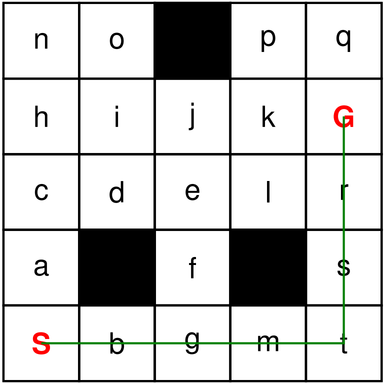

$$\tau = \{b, g, m, t, s, r\} \cup \{\neg a, \neg c, ..\}$$

$$cost = a + b + c + \ldots + r + s + t = 6$$
$$cost = \{(\neg a), (\neg b), \ldots, (\neg s), (\neg t)\}$$

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Solution Improving Search

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Intuition

1. Obtain a solution τ^*
2. Update UB
3. Improve τ^* until proven optimal

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c	d	e	l	r
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UB $= \infty$

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SAT-SOLVE($\mathcal{F}_\mathbf{H}$)

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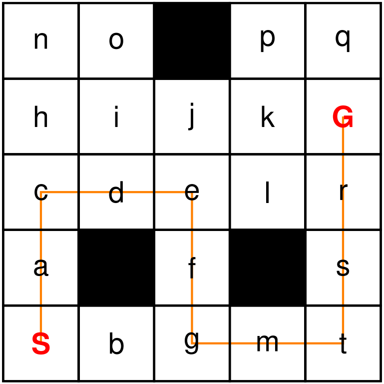

$\mathbf{UB = 10}$

SAT-SOLVE$(\mathcal{F}_H)$

$\tau^1 = \{a, c, d, e, f, g, m, u, t, r, \neg b, \neg l, \ldots, \neg q\}$

$cost(\tau^1) = 10$

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UB $= 10$

SAT-SOLVE $(\mathcal{F}_{\mathbf{H}} \wedge \textsc{CostLessThan}(\mathbf{UB}))$

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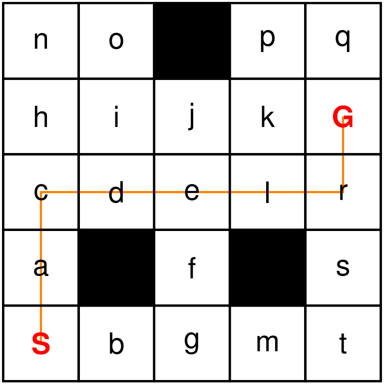

UB = 6

SAT-SOLVE $(\mathcal{F}_H \wedge \textsc{CostLessThan}(UB))$

$\tau^2 = \{a, c, d, e, l, r, \neg b, \neg g, \ldots, \neg q, \neg s, \neg t\}$
$cost(\tau^2) = 6$

Additional Techniques

Central Challenge

- Encoding CostLessThan(UB) as clauses.

- State of the art use the watchdog encoding [Paxian, Reimer, and Becker, 2018].
- Additional inference rules for decreasing the size of the objective.
  - e.g TrimMaxSAT for finding assignments of objective literals implied by the clauses [Paxian, Raiola, and Becker, 2021].

Solvers

QMaxSAT [Koshimura, Zhang, Fujita, and Hasegawa, 2012]
Pacose [Paxian, Reimer, and Becker, 2018]
Also commonly applied in *incomplete solving*.

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Core Guided Search

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First proposed for MaxSAT in [Fu and Malik, 2006]

Intuition

1. Starting from $LB = 0$ check existence of solution τ for which $\mathit{cost}(\tau) = LB$.
2. Increase LB until optimum reached by relaxing formula.
3. Use cores provided by SAT-solver for more effective relaxation.

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3. Use cores provided by SAT-solver for more effective relaxation.

Definition: UNSAT cores

- Clause (or set of) of objective literals satisfied by all solutions.
  - equivalent to an unsatisfiable set of unit soft clauses.
- Can be obtained from SAT solvers via assumptions.

n	o		p	q
h	i	j	k	**G**
c	d	e	l	r
a		f		s
S	b	g	m	t

$\kappa^1 = \{a, b\} \equiv (a \vee b) \equiv \{(\neg a), (\neg b)\}$
all paths go through either a or b

$\kappa^2 = \{h, d, f, m\}$
all paths go through (at least) one of h, d, f or m.

$\kappa^3 = \{q, k, r\}$
all paths go through (at least) one of q, k or r.

$\vdots$

Core Guided Search

Shortest path

Intuition

1. Initialise $\mathcal{F}_H^0 = \mathcal{F}_H$ and $\mathcal{F}_B^0 = \{\neg b \mid b \in cost\}$
2. For $i = 0, \ldots$ check if $\mathcal{F}_H^i \wedge \mathcal{F}_B^i$ is satisfiable
3. If not, relax $\mathcal{F}_H^i$ and $\mathcal{F}_B^i$
4. Otherwise, the obtained solution is optimal

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$\mathrm{LB} = 0$, $\mathcal{K} = \emptyset$

SAT-SOLVE($\mathcal{F}_{\mathbf{H}}^{\mathbf{i}} \wedge \mathcal{F}_{\mathbf{B}}^{\mathbf{i}}$)

Informally: $\mathcal{F}_H \wedge \bigwedge_{\kappa \in \mathcal{K}} \sum_{b \in \kappa} b \leq 1 \wedge \bigwedge_{b \notin \mathcal{K}} \neg b$

i.e. is there a path that visits at most 1 node from each found core

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c	d	e	l	r
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LB $= 0$, $\mathcal{K} = \emptyset$

SAT-SOLVE($\mathcal{F}_{\mathbf{H}}^{\mathbf{i}} \wedge \mathcal{F}_{\mathbf{B}}^{\mathbf{i}}$)

Formula is unsatisfiable

Obtain new core: $\kappa_0 = \{(\neg a), (\neg b)\}$

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LB $= 1$, $\mathcal{K} = \{\kappa_0\}$

SAT-SOLVE($\mathcal{F}_H^i \wedge \mathcal{F}_B^i$)

Formula is unsatisfiable

Obtain new core: $\kappa_1 = \{(\neg q), (\neg k), (\neg r)\}$

Core Guided Search

Shortest path

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n	o		p	q
h	i	j	k	**G**
c	d	e	l	r
a		f		s
S	b	g	m	t

LB $= 2$, $\mathcal{K} = \{\kappa_0, \kappa_1\}$

SAT-SOLVE($\mathcal{F}_{\mathbf{H}}^{\mathbf{i}} \wedge \mathcal{F}_{\mathbf{B}}^{\mathbf{i}}$)

Informally: $\mathcal{F}_H \wedge \bigwedge_{\kappa \in \mathcal{K}} \sum_{b \in \kappa} b \leq 1 \wedge \bigwedge_{b \notin \mathcal{K}} \neg b$

i.e. is there a path that visits at most 1 node from each found core

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LB $= 6$, $\mathcal{K} = \{\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}$

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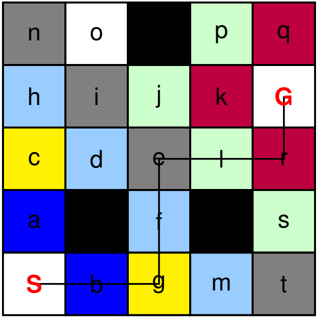


LB $= 6$, $\mathcal{K} = \{\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}$

SAT-SOLVE($\mathcal{F}_{\mathbf{H}}^{\mathbf{i}} \wedge \mathcal{F}_{\mathbf{B}}^{\mathbf{i}}$)

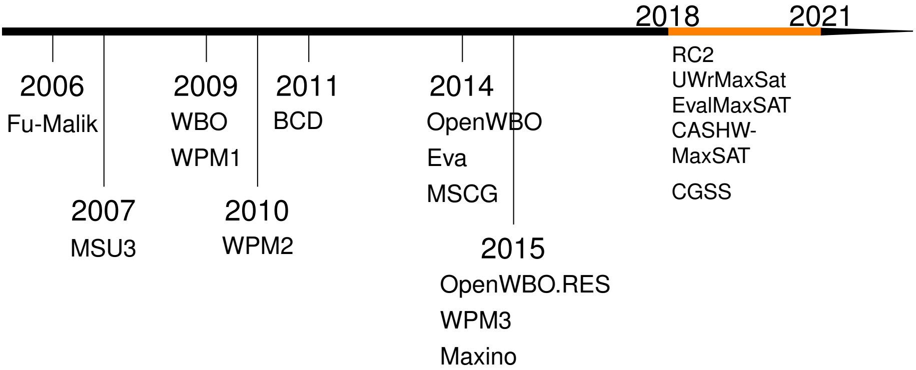
Formula is satisfiable

Obtain optimal model: $\tau = \{b, \ldots, l, r, \neg a, \ldots, \neg q\}$

$cost(\tau) = 6$

Core-Guided Algorithms

Central Developments



Core-Guided Algorithms

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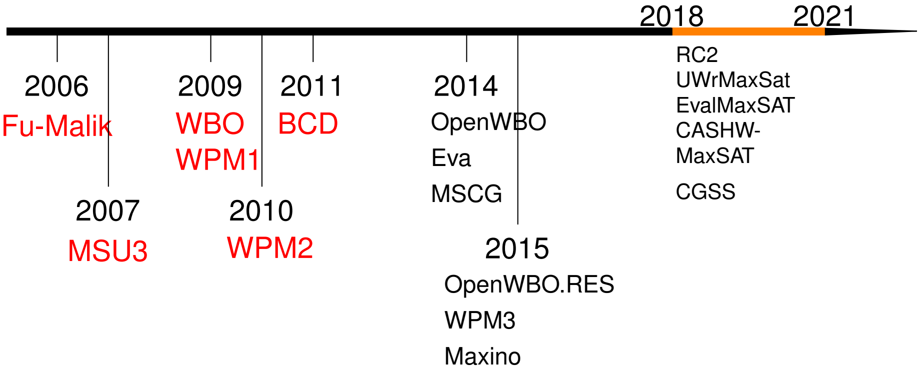

Early algorithms: Core relaxations with hard constraints

[Fu and Malik, 2006; Marques-Silva and Planes, 2007; Manquinho, Marques-Silva, and Planes, 2009; Ansótegui, Bonet, and Levy, 2009, 2010; Heras, Morgado, and Marques-Silva, 2011]

Core-Guided Algorithms

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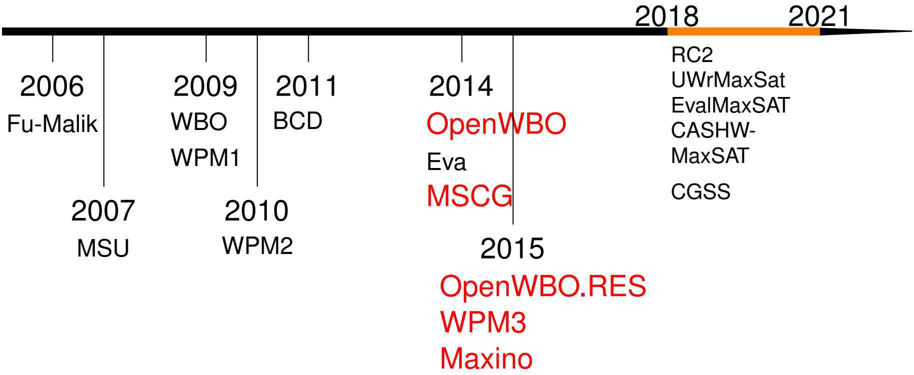


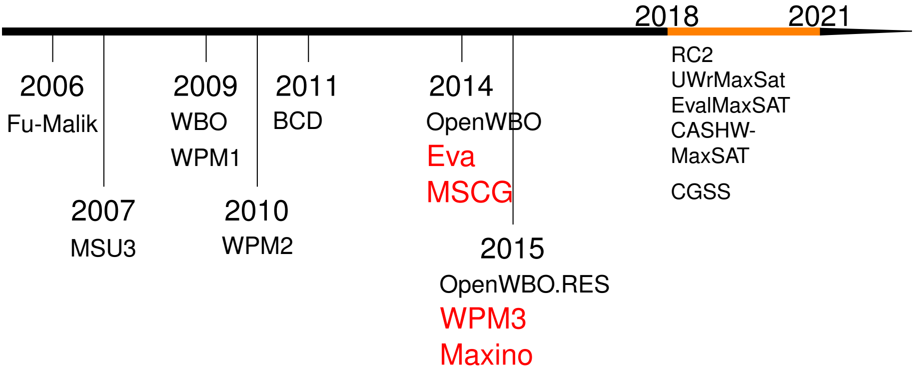
Central improvement 1: Incremental (Lazy) relaxation (cardinality) constraints

[Martins, Joshi, Manquinho, and Lynce, 2014; Morgado, Dodaro, and Marques-Silva, 2014]

Used by all current solvers

Core-Guided Algorithms

Central Developments



Central Improvement 2: Core relaxations with soft constraints

[Narodytska and Bacchus, 2014; Morgado, Dodaro, and Marques-Silva, 2014; Ansótegui, Didier, and Gabàs, 2015]

Core-Guided Algorithms

Central Developments

2006
Fu-Malik

2007
MSU3

2009
WBO
WPM1

2010
WPM2

2011
BCD

2014
OpenWBO
Eva
MSCG

2015
OpenWBO.RES
WPM3
Maxino

2018 2021
RC2
UWrMaxSat
EvalMaxSAT
CASHW-MaxSAT
CGSS

Central Improvement 2: Core relaxations with soft constraints [Narodytska and Bacchus, 2014; Morgado, Dodaro, and Marques-Silva, 2014; Ansótegui, Didier, and Gabàs, 2015]

Current state-of-the-art, effective implementations of OLL [Andres, Kaufmann, Matheis, and Schaub, 2012; Morgado, Dodaro, and Marques-Silva, 2014]

(A few) Additional Techniques

- cover optimization, structure sharing, intrinsic atmost1 [Ansótegui, Bonet, and Levy, 2010; Ansótegui, Didier, and Gabàs, 2015; Ansótegui and Gabàs, 2017; Ihalainen, Berg, and Järvisalo, 2021; Ignatiev, Morgado, and Marques-Silva, 2019]
  - Analyze underlying core structure in order to improve relaxation constraints.
- stratification, weight-aware core extraction, soft clause partitioning [Ansótegui, Bonet, Gabàs, and Levy, 2012; Berg and Järvisalo, 2017; Neves, Martins, Janota, Lynce, and Manquinho, 2015]
  - Extract cores that result in bigger lower bound increases.
  - Connections to multi-level optimization.
- hardening [Ansótegui, Bonet, Gabàs, and Levy, 2012]
  - Fix values of objective literals based on upper bounds
- Recently, presolving with an ILP solver.
  - More on this in the MSE talk.

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- stratification, weight-aware core extraction, soft clause partitioning [Ansótegui, Bonet, Gabàs, and Levy, 2012; Berg and Järvisalo, 2017; Neves, Martins, Janota, Lynce, and Manquinho, 2015]
  - Extract cores that result in bigger lower bound increases.
  - Connections to multi-level optimization.
- hardening [Ansótegui, Bonet, Gabàs, and Levy, 2012]
  - Fix values of objective literals based on upper bounds
- Recently, presolving with an ILP solver.
  - More on this in the MSE talk.

(A few) Additional Techniques

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Implicit Hitting Set Based MaxSAT Solving

Definition: Hitting sets

- Set of objective literals with non-empty intersection with cores.
- *cost* of hitting set, number of literals in it.

n	o		p	q
h	i	j	k	**G**
c	d	e	l	r
a		f		s
S	b	g	m	t

$$\kappa^1 = \{a, b\}$$

$$\kappa^2 = \{h, d, f, m\}$$

$$\kappa^3 = \{q, k, r\}$$

$$\text{CORES} = \{\kappa^1, \kappa^2, \kappa^3\}$$

$$hs_1 = \{a, d, f, q\} \quad cost(hs_1) = 4$$

$$hs_2 = \{b, m, q\} \quad cost(hs_2) = 3$$

MaxSAT with hitting sets

First proposed in [Davies and Bacchus, 2011]

Intuition

- Every solution corresponds to a hitting set over all cores.
- Cost of solutions match cost of corresponding hitting sets.
- Central insight - we do not need every core.
  - $cost(hs) \leq$ OPT-COST($\mathcal{F}$) holds for minimum cost hs over *any* set of CORES

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$\kappa^1 = \{a, b\}$

$\kappa^2 = \{h, d, f, m\}$

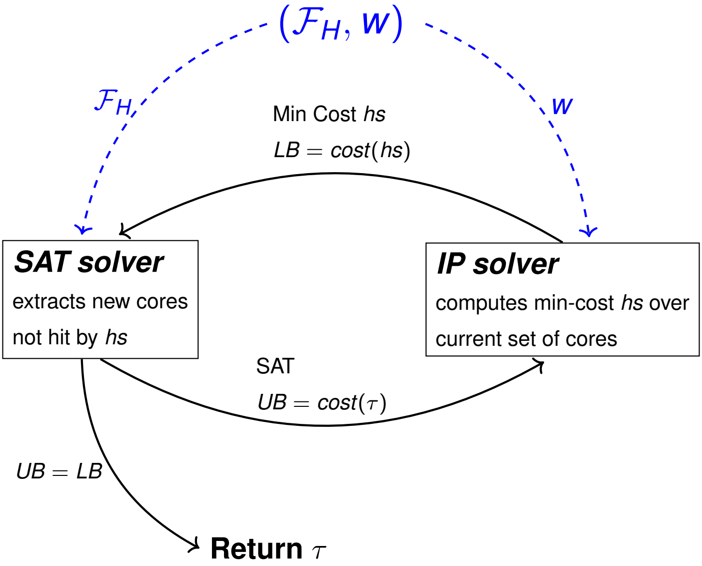
$\kappa^3 = \{q, k, r\}$

$\textsc{cores} = \{\kappa^1, \kappa^2, \kappa^3\}$

$hs = \{b, m, q\} \quad cost(hs) = 3 \leq 6 = \textsc{opt-cost}(\mathcal{F})$

The IHS approach to MaxSAT

[Davies and Bacchus, 2011]



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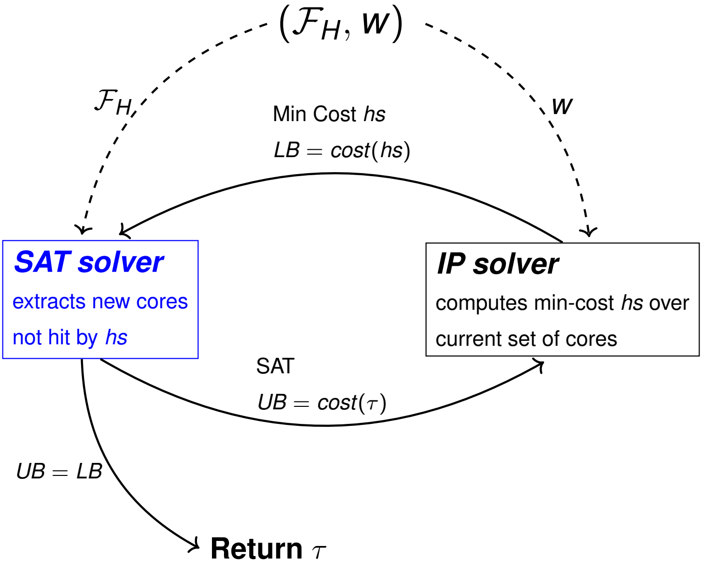

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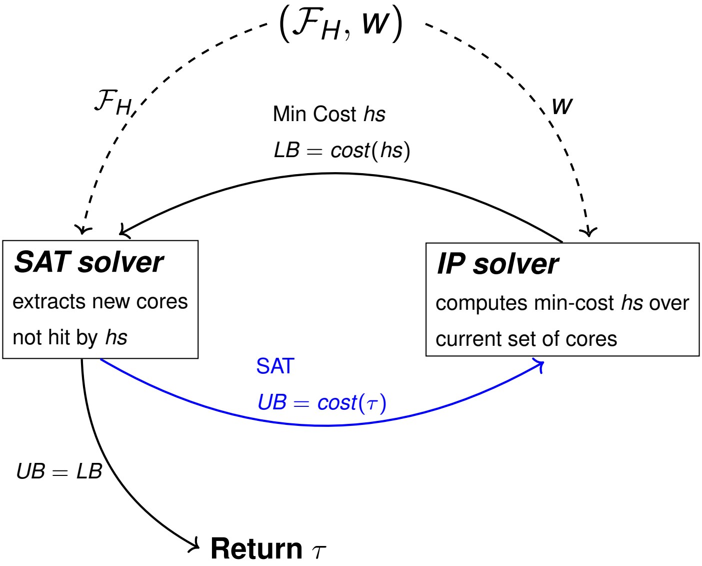

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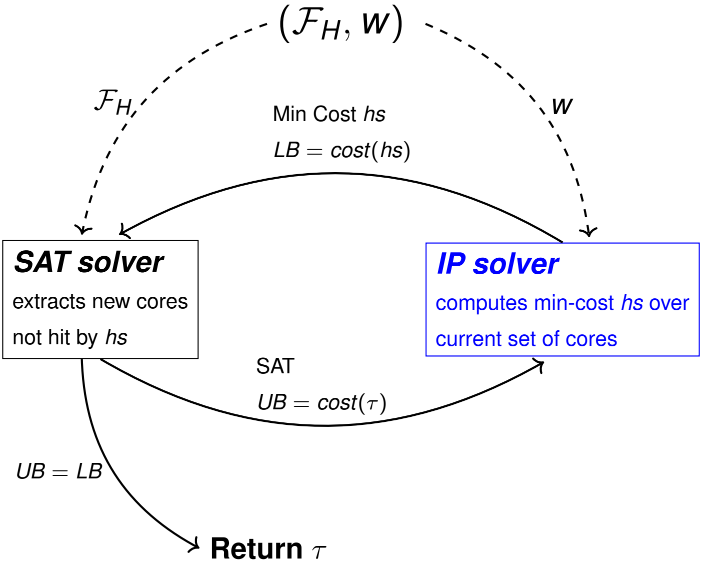

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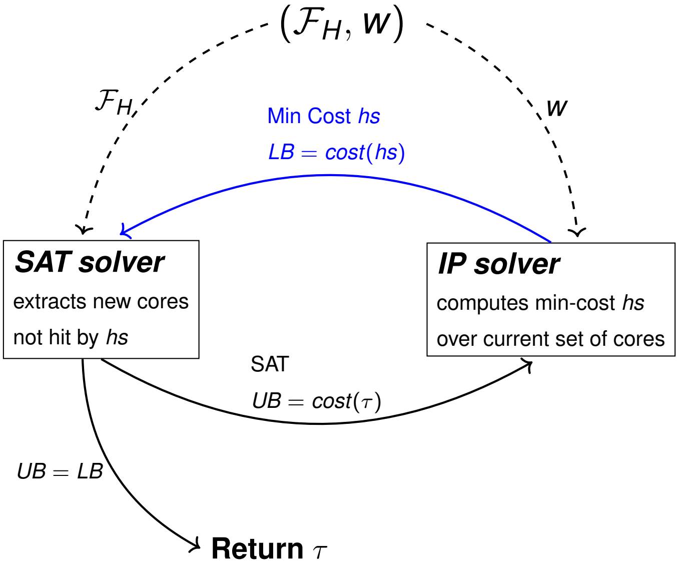

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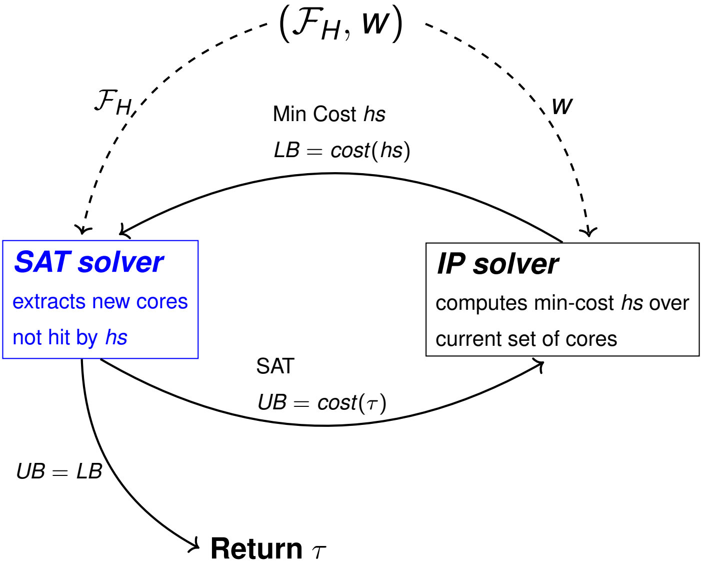

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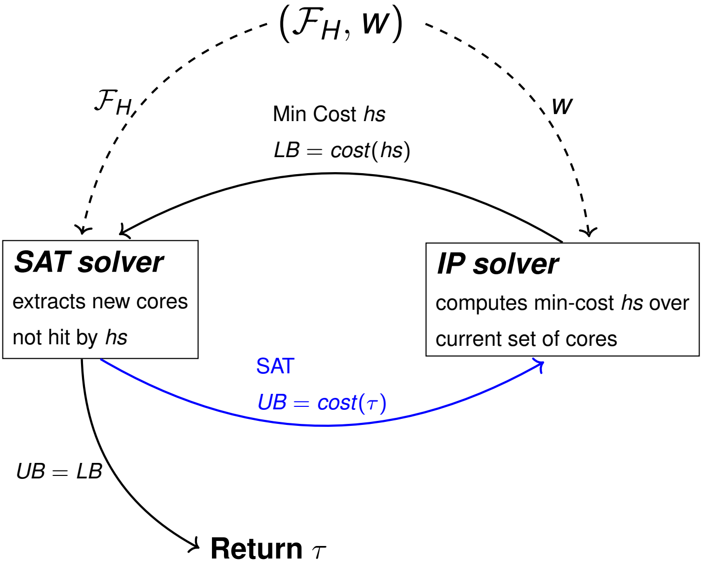

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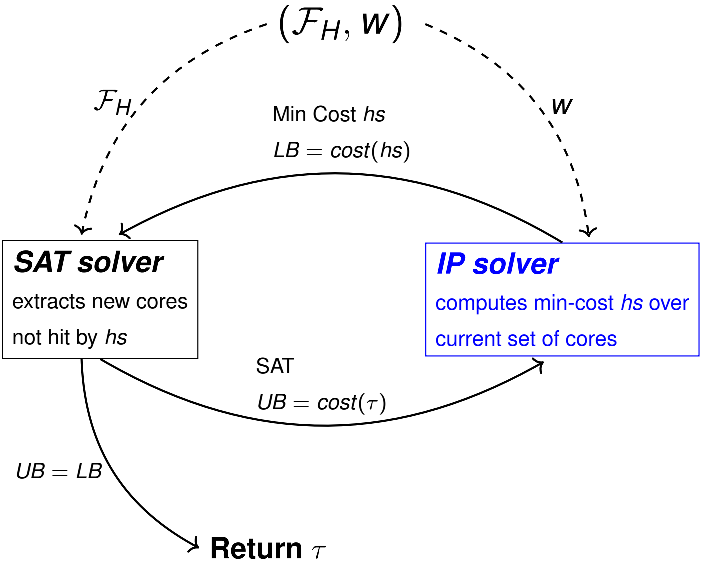

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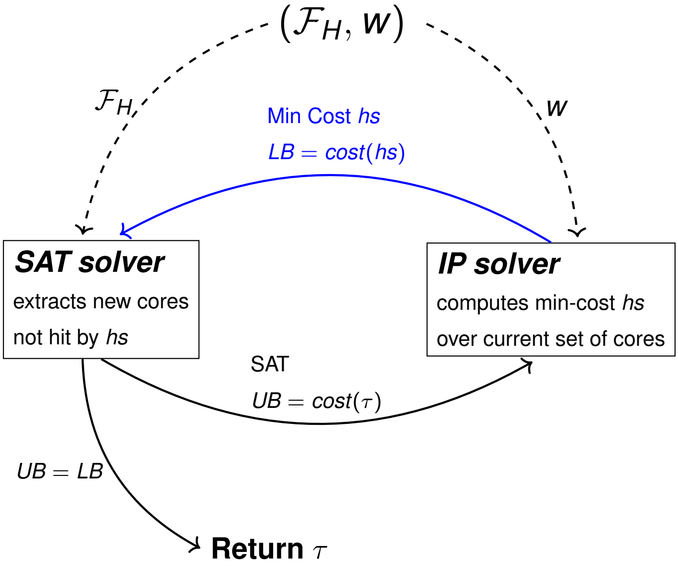

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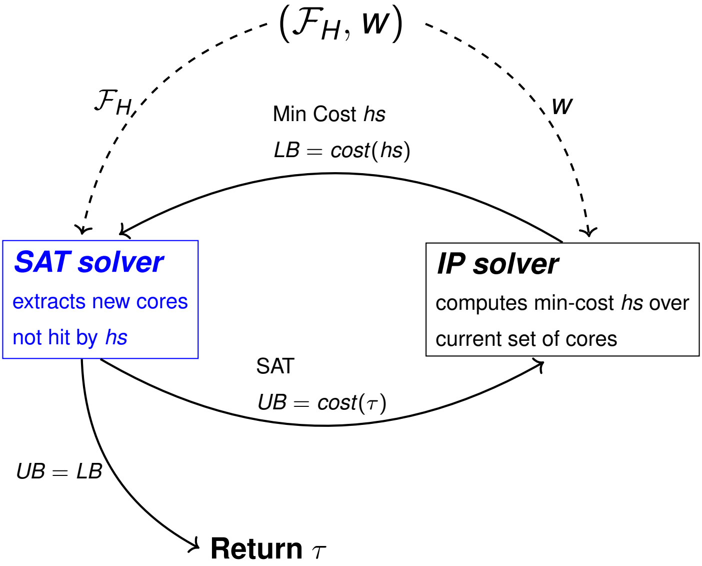

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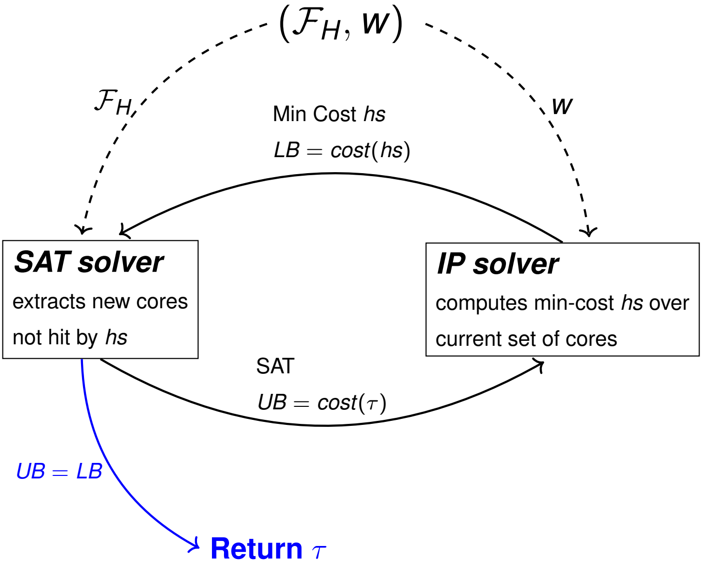

(Some) Further Techniques

- non-optimal hitting sets [Saikko, Berg, and Järvisalo, 2016].
  - extract more cores.
- reduced cost fixing [Bacchus, Hyttinen, Järvisalo, and Saikko, 2017]
  - use information from the IP solver in order to fix objective literals in the SAT-solver
- abstract cores [Berg, Bacchus, and Poole, 2020].
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Solvers

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MaxHS [Davies and Bacchus, 2013],
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Algorithms

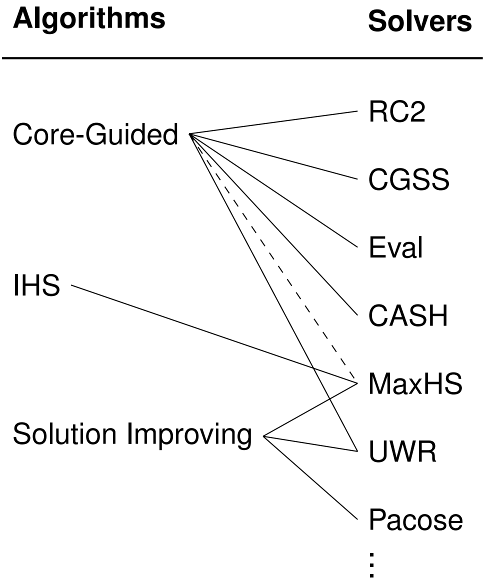
Core-Guided

IHS

Solution Improving

SAT-Based MaxSAT solving

Conclusion



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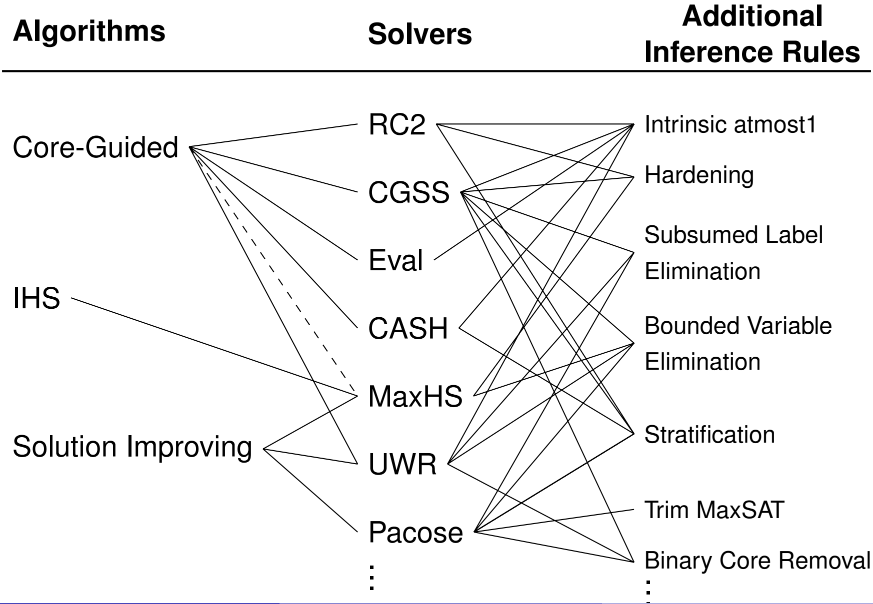

Take Home Message
Modern SAT-Based MaxSAT solvers implement a large number of different heuristics and algorithms that interact in intricate ways.

A central challenge
of the field is understanding these interactions and which techniques are effective on the benchmarks of interest.

Other Interesting Topics

Other Research Directions

Incremental MaxSAT

- Solve sequences of related MaxSAT instances.
- IPAMIR interface for incremental computations.
  - Used to realize the new incremental track in this years MSE.

[Niskanen, Berg, and Järvisalo, 2022; Si, Zhang, Manquinho, Janota, Ignatiev, and Naik, 2016; Niskanen, Berg, and Järvisalo, 2021]

Incomplete (any-time) MaxSAT

- Compute a solution of as low cost as possible within limited time & memory.
- Many specialised solvers, including local search solvers.

[Cohen, Nadel, and Ryvchin, 2021; Zheng, He, Zhou, Jin, Li, and Manyà, 2022; Cai and Lei, 2020; Nadel, 2018; Berg, Demirovic, and Stuckey, 2019; Demirovic and Stuckey, 2019]

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Other Research Directions

Branch & Bound for MaxSAT

- Early work effective especially on small, difficult problems. [Abramé and Habet, 2016, 2014; Li, Manyà, and Planes, 2005]
- Recent paper on adding clause learning to B&B algorithms.
  - MaxCDCL solver [Li, Xu, Coll, Manyà, Habet, and He, 2021].

Abstract Reasoning & Preprocessing

- MaxSAT resolution [Bonet, Levy, and Manyà, 2007; Bonet, Buss, Ignatiev, Marques-Silva, and Morgado, 2018]
- Clause redundancy notions lifted from SAT to MaxSAT [Ihalainen, Berg, and Järvisalo, 2022; Belov, Morgado, and Marques-Silva, 2013].
- Standalone preprocessors available.
  - MaxPRE [Korhonen, Berg, Saikko, and Järvisalo, 2017; Ihalainen, Berg, and Järvisalo, 2022].
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Conclusions

SAT-based MaxSAT solving:

- Effective approach for solving industrial MaxSAT instances
- Combine various algorithms and heuristics in non trivial ways.
- Active (and interesting) area of research.
  - Many ideas have also been studied in e.g. PBO and CP.
    [Devriendt, Gocht, Demirovic, Nordström, and Stuckey, 2021; Smirnov, Berg, and Järvisalo, 2021; Gange, Berg, Demirovic, and Stuckey, 2020]

Further Resources

Surveys in the handbook of satisfiability [Bacchus, Järvisalo, and Martins, 2021; Li and Manyà, 2021]

The webpage of the MaxSAT Evaluation:
https://maxsat-evaluations.github.io/.

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